The OSCAR PUBLISHING Services
American Journal of Applied Science and Technology

VOLUME Vol.05 Issue 10 2025
PAGE NO. 218-224
DOI 10.37547/ajast/Volume05Issue10-38

Numerical Modeling Of Reaction–Diffusion Processes Described By Doubly Nonlinear Parabolic Equations With A Source Term

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Received: 30 August 2025; **Accepted:** 24 September 2025; **Published:** 29 October 2025

Abstract: In this paper, reaction-diffusion processes described by nonlinear parabolic equations with a source term are modeled. Using a self-similar (automodel) approach, the differential equation is simplified, and an initial approximation is constructed to obtain a numerical solution. During the computation, an appropriate difference scheme is selected, and the stability and convergence properties are analyzed. The results obtained through computer simulations are presented graphically and evaluated in terms of how well they reflect the dynamics of the physical processes. The proposed approach is shown to be effective for obtaining numerical solutions to such models.

Keywords: Doubly nonlinear parabolic equations; reaction–diffusion processes; numerical modeling; finite difference method; nonlinear dynamics; source term; stability analysis; convergence; diffusion–reaction systems; computational mathematics; nonlinear PDEs; mathematical modeling.

INTRODUCTION:

Below, on the domain $Q=\{(t,x),t>0,x\in R\}$

$$|x|^{-q}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(|x|^{n}\left|\frac{\partial u^{k}}{\partial x}\right|^{p-2}\frac{\partial u^{m}}{\partial x}\right)+|x|^{-q}u^{q_1} \quad (1)$$

$$u_{t=0}=u_0(x)\geq 0,x\in R \quad (2)$$

We consider the Cauchy problem for a divergence-form doubly nonlinear parabolic equation with a source and variable density, where q,m,n,k,p,q_1 are numerical parameters characterizing the nonlinear medium, and $u(t,x)\geq 0$ is the unknown solution.

Equation (1) describes nonlinear heat conduction, diffusion, filtration, biological population dynamics, and various other processes [1–10].

(1) is in divergence form; therefore, in regions where $u=0$ or $\nabla u=0$ the equation becomes degenerate and may fail to have a classical solution. In such cases one seeks and studies weak (generalized) solutions of (1). Accordingly, we regard its solution as a generalized one in the class

$$0 \le u, \left|\frac{\partial u^k}{\partial x}\right|^{p-2} \frac{\partial u^m}{\partial x} \in C(Q)$$

and (1) is satisfied in the sense of an integral identity.

For $m=1, p=2, q=0$ equation (1) was proposed by L. Leibenson [16] to describe oil-and-gas processes. For $p=2, q_1=0, q=0$ it models phenomena in a porous medium and other processes.. For $q_1=0$ it represents the Hamilton–Jacobi equation [1–5].

(1), (2) The existence and uniqueness of solutions to the Cauchy problem for various specific values of the parameters have been studied in [6–20]. In the work of A. A. Samarskii and I. M. Sobol' [15], numerical solutions were considered for the parameter values $m=1, p=2, q=0$. In papers by H. Fujita, Daniela Giachetti, and M. Michaela Porzio, the global solvability of the Cauchy problem for nonlinear parabolic equations was investigated. In Huashui Zhan's work, properties of solutions to the nonlinear diffusion equation with a damping term were studied. The non-divergence case of problems (1), (2) can be found in [20].

METHODOLOGY

We seek the solution $u(t,x)$ in the form:

$$u(t,x) = \bar{u}(t) z(\tau(t), \varphi(|x|)).$$

where $\bar{u}(t)$, $\tau(t)$ and $\varphi(|x|)$ are defined as below, and

$$\bar{u}(t) = (T + (1-q_1)t)^{\frac{1}{1-q_1}},$$

$$\tau(t) = \frac{\bar{u}^{m+k(p-2)-q_1}(t)}{m+k(p-2)-q_1}$$

we defined in the following form:

$$\varphi(x) = \begin{cases} \frac{p}{p-q-n}|x|^{\frac{p-q-n}{p}}, \text{if } p \ne q+n \\ \ln|x| \quad , \text{if } p = q+n \end{cases} \tag{3}$$

Then we reduced equation (1) to the following "radially symmetric" form.

$$\frac{\partial z}{\partial t} = \varphi^{1-s} \frac{\partial}{\partial \varphi}\left(\varphi^{s-1}\left|\frac{\partial z^k}{\partial \varphi}\right|^{p-2} \frac{\partial z^m}{\partial \varphi}\right) + \frac{1}{\tau(m+k(p-2)-q_1)}(z^{q_1} - z) \tag{4}$$

(4) we reduce the equation to a self-similar equation by the substitution

$z(\tau(t), \varphi(|x|)) = f(\xi),\ \xi = \frac{\varphi}{\tau^{\frac{1}{p}}},\ s = \frac{p(1-q)}{p-q-n}$ here ξ is the self-similar variable and $f(\xi)$ is the self-similar function.

$$-\frac{\xi}{p}\frac{df}{d\xi} = \xi^{s-1}\frac{d}{d\xi}\left(\xi^{1-s}\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) + \frac{1}{(m+k(p-2)-q_1)}\left(f^{q_1} - f\right). \tag{5}$$

Below, we consider the nontrivial, nonnegative solutions of equation (5) that satisfy the following conditions:

$$f(0) = M,\ M \in R,\ f(d) = 0,\ 0 < d < \infty \tag{6}$$

Next, using the method of comparing solutions [1] and the method of model (reference) equations [3], we obtain estimates for the solution of problem (1), (2).

Theorem 1. Suppose the inequality $q_1 < m + k(p-2)$ holds, and one of the inequalities

$$\begin{aligned} &1)\, q < 1,\, p > n+1 \\ &2)\, q > 1,\, p < n+1 \\ &3)\, q < 1,\, p < n+q \\ &4)\, q > 1,\, p > n+q \end{aligned} \tag{7}$$

And suppose $u(0,x) \le u_+(0,x), x \in R$. Then problem (1), (2) has a global solution in the domain Q and the following upper estimate holds:

$$u(t,x) \le u_+(t,x)$$

where:

$$u_+(t,x) = A\left(a - \left(\frac{\varphi(x)}{\tau^{\frac{1}{p}}}\right)^{\gamma_1}\right)^{\gamma_2} \cdot \bar{u}(t),$$

$$A = \left(\frac{m+p+k(p-2)-1}{p\left(pk^{p-2}\right)^{\frac{1}{p-1}}}\right)^{m+k(p-2)-1},\quad a = const > 0,\ \gamma_1 = \frac{p}{p-1},\ \gamma_2 = \frac{p-1}{m+k(p-2)-1}.$$

Proof. We represent $L(u_+(t,x))$ as the sum $L(u_+(t,x)) = L_1(u_+(t,x)) + L_2(u_+(t,x))$ and prove Theorem 1. Here

$$L_1(u_+(t,x)) = \frac{d}{d\xi}\left(\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) + \frac{1}{p}\frac{d}{d\xi}(\xi \cdot f)$$

$$L_2(u_+(t,x)) = \frac{s-1}{\xi}\frac{d}{d\xi}\left(\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) + \frac{1}{(m+k(p-2)-q_1)}\left(f^{q_1}-f\right) - \frac{f}{p}$$

We will show that $L(f(\xi)) \geq 0$ when $|\xi| < a^{\frac{p-1}{p}}$ To do this, we proceed as follows: first, we calculate $L_1(u_+(t,x)) = 0$ and then prove that $L_2(u_+(t,x)) \geq 0$

Now let us consider the following equation:

$$L_1(u_+(t,x)) = \frac{d}{d\xi}\left(\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) + \frac{1}{p}\frac{d}{d\xi}(\xi \cdot f) = 0$$

The solution of this differential equation is as follows:

$$f(\xi) = A\left(a - \xi^{\frac{p}{p-1}}\right)^{\frac{p-1}{m+k(p-2)-1}}.$$

Now we prove that, under the assumptions of Theorem 1, the inequality $L_2(u_+(t,x)) \geq 0$ holds. Indeed,

$$L_2(u_+(t,x)) = \frac{s-1}{\xi}\frac{d}{d\xi}\left(\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) + \frac{1}{(m+k(p-2)-q_1)}\left(f^{q_1}-f\right) - \frac{f}{p} \quad (8)$$

Since $L_1(u_+(t,x)) = 0$ we can rewrite $L_1(u_+(t,x))$ as:

$$\frac{d}{d\xi}\left(\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df^m}{d\xi}\right) = -\frac{1}{p}\frac{d}{d\xi}(\xi \cdot f).$$

Substituting this relation into $L_2(u_+(t,x))$ gives:

$$f\left(-\frac{s}{p} - \frac{1}{m+k(p-2)-q_1}\right) + \frac{f^{q_1}}{(m+k(p-2)-q_1)} = 0.$$

If $\beta < m+k(p-2)$ and one of the inequalities (2) or (7) is satisfied, then the following inequality holds.

$$-f\left(\frac{1-q}{(p-n-q)} + \frac{1}{m+k(p-2)-q_1}\right) + \frac{f^{q_1}}{m+k(p-2)-q_1} \geq 0$$

Relying on the comparison theorem, we showed that the estimate $u(t,x) \leq u_+(t,x)$ is valid in the domain Q. Therefore, Theorem 1 is proved.

From problem (1) we obtain the following one-dimensional nonlinear heat-diffusion equation with initial and boundary conditions.

(1) masaladan biz boshlang'ich va chegaraviy shartlarga ega bo'lgan bir o'lchovli chiziqli bo'lmagan quyidagi issiqlik tarqalish tenglamasiga egamiz: For problems (1) and (2), we construct a mesh on the number axes in x and τ with steps $h,\ \tau$ respectively:

$$\bar{w}_h = \{x_i = ih, h>0, i=\overline{0,n}, hn=b\},\ \bar{w}_\tau = \{t_j = j\tau, \tau>0, j=\overline{0,m}, \tau m = T\}$$

With an error of $O(h^2+\tau)$ we replace problem (1) by a finite-difference equation using an implicit scheme:

$$\begin{cases} \dfrac{y_i^{j+1}-y_i^j}{\tau} = \dfrac{1}{h^2}\left[a_{i+1}(y^j)(y_{i+1}^{j+1}-y_i^{j+1}) - a_i(y^j)(y_i^{j+1}-y_{i-1}^{j+1})\right] + (y_i^j)^{q_1} \\ i=1,2,\dots,n-1;\ j=0,1,\dots,m-1 \\ y_i^0 = u_0(x_i),\ \ i=0,1,2,\dots,n \\ y_0^j = \varphi_1(\tau_j),\ j=1,2,\dots,m \\ y_n^j = \varphi_2(\tau_j),\ j=1,2,\dots,m \end{cases} \tag{9}$$

here a_{i+1} and a_i are the nonlinear terms of problem (9). In our case a_{i+1} and a_i are the heat-conductivity coefficients. We compute a_{i+1} and a_i using the following formula:

$$a_{i+1}(y^j) = m\cdot|x_i|^{n+q}\cdot(y_i^j)^{m-1}\left|\frac{(y_{i+1}^j)^k-(y_i^j)^k}{h}\right|^{p-2};$$

$$a_i(y^j) = m\cdot|x_i|^q\cdot|x_{i-1}|^n\cdot(y_{i-1}^j)^{m-1}\left|\frac{(y_i^j)^k-(y_{i-1}^j)^k}{h}\right|^{p-2} \tag{9}$$

The system is a system of **nonlinear algebraic equations** with respect to y^{j+1} .To solve it, we use the simple iteration method and reduce it to the following linear system of algebraic equations:

$$\frac{\overset{s+1}{y}{}_i^{j+1} - \overset{s}{y}{}_i^j}{\tau} = \frac{1}{h^2}\left[a_{i+1}\left(\overset{s}{y}{}^j\right)\left(\overset{s+1}{y}{}_{i+1}^{j+1} - \overset{s+1}{y}{}_i^{j+1}\right) - a_i\left(\overset{s}{y}{}^j\right)\left(\overset{s+1}{y}{}_i^{j+1} - \overset{s+1}{y}{}_{i-1}^{j+1}\right)\right] + \left(\overset{s}{y}{}_i^j\right)^{q_1} \tag{10}$$

Here $s=0,1,\dots$; is the iteration number. The iteration process continues until the condition

$$\max\left|\overset{s+1}{y}_i - \overset{s}{y}_i\right| < \varepsilon,\ 0\le i\le n.$$

is satisfied.

Remark. In all numerical calculations we take $\varepsilon = 10^{-3}$. Introduce the notation

$$y^j = y,\ y^{j+1} = \bar{y}, \quad (11)$$

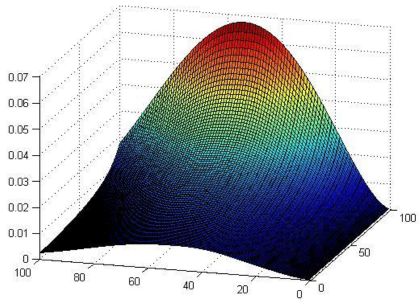
From scheme (11) we obtain the coefficients of the tridiagonal matrix A_i, B_i, C_i, F_i and, using the sweep (Thomas) method, solve the following linear system:

$$A_i \bar{y}_{i-1} - C_i \bar{y}_i + B_i \bar{y}_{i+1} = -F_i,\ i = 1,2,\ldots,n-1.$$

$$y = \lambda_1 y_1 + \mu_2,\ y_N = \lambda_2 y_{N-1} + \mu_2.$$

RESULTS AND DISCUSSION

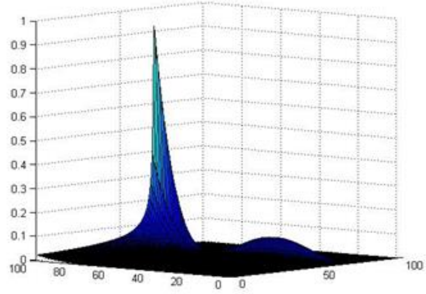
In the problem under consideration, the following nonlinear effects are observed: temporal unboundedness (blow-up) of the solution; the phenomenon of finite speed of heat propagation and its spatial localization; and, under the influence of a source term, the presence of finite-time displacement in nonlinear media.



$m = 1.23;\ k = 1.5;\ p = 2.29$

$n = 1.4, q = 1.3; q_1 = 1.7$

Figure 1. Reverse (backward) diffusion case $m + k(p-2) - 1 < 0$



$m = 1.26;\ k = 1.5;\ p = 2.27$

$n = 1.6, q = 1.42; q_1 = 1.9$

Figure 2. Slow diffusion case $m + k(p-2) - 1 > 0$

CONCLUSION AND RECOMMENDATIONS

In this article, the numerical modeling of doubly nonlinear parabolic equations with a source term describing reaction–diffusion processes was carried out. Using a self-similar (automodel) approach, the equation was simplified and an initial approximation for obtaining a numerical solution was determined. An appropriate difference scheme was chosen, a computational algorithm was developed, and stability conditions were analyzed. The results obtained by computer were presented graphically, demonstrating the model's consistency with the physical essence of the problem. The findings confirm that the chosen method is effective and reliable for solving complex problems of the reaction–diffusion type.

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