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# Bessel functions of the first kind

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**Abstract:** This paper discusses the derivation of Bessel functions of the first kind using power series method and their properties. Additionally, the practical applications of these functions, their graphical analysis, and relationships with other special functions are examined. The research results serve to expand the theoretical and practical significance of Bessel functions.

**Keywords:** Bessel functions, power series, mathematical analysis, physical phenomena, engineering problems, special functions.

**Introduction:** Bessel functions are widely used in problems with cylindrical symmetry, particularly in the analysis of vibrations, heat transfer, and electromagnetic waves. These functions have found their significant place in numerous physical phenomena and engineering problems, especially in the analysis of oscillations, heat conduction, and electromagnetic wave propagation. Moreover, their graphical analysis and application through mathematical formulas provide more effective solutions to physical problems.

The paper illustrates the derivation of Bessel functions of the first kind using power series method and related mathematical properties. Furthermore, it analyzes these functions' applications in physical problems, their practical significance, and relationships with other special functions. The research results contribute to the broader application of Bessel functions and expand their importance in scientific research, engineering, and physics.

Bessel functions are typically defined through differential equations and are derived using power series. The Bessel function of the first kind is written as:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left(\frac{x}{2}\right)^{2n+v}}{n! \cdot \Gamma(v+n+1)}$$

where $J_n(x)-$ is the Bessel function, $x-$ is the argument, $v-$ is the order index, and $\Gamma-$ is the Gamma function.

The derivation of Bessel functions is accomplished through the Bessel equation:

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0 \qquad (1)$$

Or

$$y^{"}+\frac{y^{'}}{x}+\left(1-\frac{v^{2}}{x}\right)y=0 \qquad (2)$$

this equation is called the Bessel equation, where the constant $v-$ is called the index of the equation.

In the next step, we introduce the substitution $v\geq 0$ into the equation $y=x^{v}z$.

$$y'=\left(x^{v}\cdot z\right)^{'}=v\cdot x^{v-1}\cdot z+x^{v}\cdot z'$$

$$y''=\left(v\cdot x^{v-1}\cdot z+x^{v}\cdot z'\right)^{'}=v\cdot(v-1)\cdot x^{v-2}\cdot z+2\cdot v\cdot x^{v-1}\cdot z'+x^{v}\cdot z''$$

$$y^{"}+\frac{y^{'}}{x}+\left(1-\frac{v^{2}}{x}\right)y=0$$

$$v\cdot(v-1)\cdot x^{v-2}\cdot z+2\cdot v\cdot x^{v-1}\cdot z'+x^{v}\cdot z''+\frac{vx^{v-1}\cdot z+x^{v}\cdot z'}{x}+\left(1-\frac{v^{2}}{x}\right)\cdot x^{v}z=0$$ Simplifying

this expression and to simplify the function, we obtain the equation $z$:

$$z^{"}+\frac{(2v+1)}{x}\cdot z'+z=0 \quad (2)$$

We seek the solution of this equation in the form of: $z=\sum_{n=0}^{\infty}c_{n}x^{n}$.

$$z'=c_{1}+2c_{2}x+3c_{3}x^{2}+\ldots+(n+2)c_{x+2}x^{n+1}+\ldots$$

$$\frac{z'}{x}=\frac{c_{1}}{x}+2c_{2}+3c_{3}x+\ldots+(n+2)c_{x+2}x^{n}+\ldots$$

$$z''=2c_{2}+2\cdot 3c_{3}x^{2}+3\cdot 4c_{4}x^{2}+\ldots+(n+1)(n+2)c_{x+2}x^{n}+\ldots$$

Substituting the resulting series into the equation, we obtain the following equality:

$$\frac{2v+1}{x}c_{1}+[2c_{2}+(2v+1)2c_{2}+c_{0}]+[2\cdot 3c_{3}+(2v+1)3c_{3}+c_{1}]x+[3\cdot 4c_{4}+$$
$$(2v+1)4c_{4}+c_{2}]\cdot x^{2}+\ldots+[(n+1)(n+2)c_{n+2}+(2v+1)(n+2)c_{n+2}+c_{n}]x^{n}+\ldots=0$$

According to the method of undetermined coefficients, we set all coefficients of powers of $x$ equal to zero. This is because for the series sum to be zero, each coefficient must be zero.

$$x^{0} \text{ for } c_{1}=0 \qquad (3)$$

$$(n+1)(n+2)c_{n+2}+(2v+1)(n+2)c_{n+2}+c_{n}=0 \qquad n=1,2,\ldots$$

From this, we obtain the following recurrence formula:

$$c_{n+2}=-\frac{c_{n}}{(n+2)(n+2v+2)} \qquad n=0,\ 1,\ 2,\ldots \quad (4)$$

Based on $(3)$ and $(4)$:

$$c_1 = c_3 = c_5 = \ldots = c_{2n-1} = \ldots = 0$$

$$c_2 = -\frac{c_0}{2(2v+2)}$$

$$c_4 = -\frac{c_2}{4(2v+4)} = \frac{c_0}{2\cdot 4(2v+2)(2v+4)}$$

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$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4\cdot \ldots \cdot 2n\cdot (2v+2)(2v+4)\cdot \ldots \cdot (2v+2n)} =$$
$$= (-1)^n \frac{c_0}{2^{2n}\cdot 1\cdot 2\cdot \ldots \cdot n\cdot (v+1)(v+2)\cdot \ldots \cdot (v+n)}$$

Thus, the solution of the equation is represented by the series:

$$z = c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}\cdot 1\cdot 2\cdot \ldots \cdot n\cdot (v+1)(v+2)\cdot \ldots \cdot (v+n)}\right] \qquad (5)$$

Usually, the constant $c_0$ is chosen as:

$$c_0 = \frac{1}{2^v \Gamma(n+1)}$$

The Gamma function is considered a generalization of the factorial:

$$1\cdot 2\cdot \ldots \cdot n = n! = \Gamma(n+1)$$

$$(v+1)(v+2)\cdot \ldots \cdot (v+n)\Gamma(v+1) = (v+2)(v+3)\cdot \ldots \cdot (v+n)\Gamma(v+2) =$$
$$= (v+n)\Gamma(v+n) = \Gamma(v+n+1)$$

We can write the function in the following form:

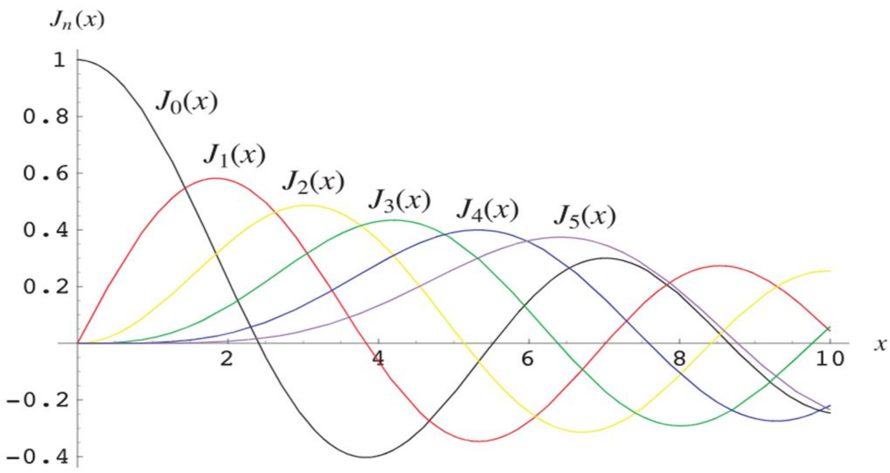
$$z = \frac{1}{2^v \Gamma(n+1)} + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+v}\cdot 1\cdot 2\cdot \ldots \cdot n\cdot (v+1)(v+2)\cdot \ldots \cdot (v+n)\Gamma(v+1)} =$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+v}\Gamma(n+1)\Gamma(v+n+1)}$$

The solution of the equation consists of the function. We denote this function as $J_v(x)$ . Therefore,

$$J_v(x) = \sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n+v}}{\Gamma(n+1)\Gamma(v+n+1)}$$

This function is called the Bessel function of the first kind of index $v$ or order $v$ In some literature, these functions are also referred to as cylindrical functions.

The Bessel functions exhibit sinusoidal-like behavior. $J_v(x)$ has periodic oscillations similar to sine and cosine, but these oscillations gradually increase and decrease in amplitude.




1-rasm. Birinchi turdagi Bessel funksiyasi grafigi

The above graph shows Bessel functions of the first kind $J_n(x)$ for different $n = 0, 1, 2, 3, 4, 5$ values. The important characteristics of this graph are reflected in mathematical, physical, and engineering applications. As evident from the graph, the amplitude (maximum value) of each $J_n(x)$ function decreases as $x-$ increases. This phenomenon reflects the relationship of Bessel functions to resonance systems in physics and engineering.

**CONCLUSION**

Bessel functions play a crucial role in mathematical analysis and physics. Their derivation using power series method and fundamental properties make them widely applicable in solving numerous theoretical and practical problems. The graphical analysis of Bessel functions and their relationships with other special functions enable the development of new research and practical applications. Bessel functions maintain their significance in mathematical analysis and physics.

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