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Research Article

ON THE FORMULATION OF BOUNDARY VALUE PROBLEMS FOR ONE SECOND-ORDER EQUATION

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ABSTRACT

This paper addresses the formulation of boundary value problems (BVPs) for second-order differential equations. Boundary value problems are essential in various scientific and engineering applications where solutions must satisfy specific conditions at the boundaries of the domain. The study outlines a systematic approach to defining the differential equation, determining the domain, and specifying the appropriate boundary conditions. The discussion includes different types of boundary conditions such as Dirichlet, Neumann, and mixed conditions. An example is provided to illustrate the formulation process, demonstrating how to combine the differential equation with boundary conditions to define a complete BVP. Methods for solving these problems, including analytical and numerical techniques, are also reviewed, highlighting their importance in obtaining accurate solutions for complex systems.

KEYWORDS

Boundary Value Problems (BVPs), Second-order Differential Equations, Dirichlet Boundary Condition, Neumann Boundary Condition, Mixed Boundary Condition, Analytical Methods.

INTRODUCTIO

Consider the equation:

$$LU = K(x;y)U_{yy} + U_{xx} + \alpha(x;y)U_y + b(x;y)U + m|u|^{\rho}U = f(x;y) \qquad (1)$$

where K(x,y)-continuously differentiable function, and $K(x,y) \geq 0$ at $y \geq 0, K(x,y) < 0$ at $y < 0,\ a(x,y) \in \bar{C}(D), b(x,y) \in C^1(\bar{D}),\ m < 0,\ p > 0$

The region D-which consists at y>0 of a rectangle with vertices at points A(0;0), B(1;0), A_1(0;1), B_1(1;1), and at $y < 0$ is bounded by the characteristics of equation (1)

$$S_1 = \left\{(x;y) : \frac{dx}{dy} = -\sqrt{-K}, y(0) = 0, y < 0\right\}$$

$$S_2 = \left\{(x;y) : \frac{dx}{dy} = -\sqrt{-K}, y(0) = 0, y < 0\right\}$$

Let's put $S = S_1 \cup S_2$

Boundary value problem. Find a solution of equation (1) in the region D such that

$$U(0;y) = U(1;y) = 0 \qquad (2)$$

$$U(x;1) = \beta(x)\, U(x;y)/_S \qquad (3)$$

Everywhere below it is assumed that, y∈S. $\beta(x) = \exp\left[\frac{\lambda}{p+2}(-1+y)\right], \lambda > 0, y \in S.$

Where y>0 Let $W_2^1(D)$ denote the space of functions from $W_2^1(D)$ that satisfy the boundary conditions (2)-(3)

Definition 1. The functions $(x;y) \in W_2^1(D)$) are called the generalized solution of the problem (1)-(3), if the integral identity holds.

$$\int_d \left[-U_y(KV)_Y - U_xV_x + a(x;y)U_yV + bUY + m|U|^{\rho}UV\right]dD = \int_D fVdD$$

for any function V from $\overset{0}{W}{}_2^1(D)$.

The existence of a generalized solution to the boundary value problem (1)-(3) will be established using the Galerkin method. Let $\{\varphi_n(x,y)\}$ – be the set of functions from the space $W_2^1(D)$ possessing the property that all elements of $\varphi_n(x,y)$ are linearly independent, and their linear combinations are dense in this space. Such a set, as known from [1], [6] exists.

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Let's consider an auxiliary problem

$$lW_n = e^{-\lambda y} W_{ny}(x;y) = \varphi_n(x;y) \qquad (5)$$

$$W_n(x;1) = \beta(x) W_n(x;y) \Big/_S \qquad (6)$$

The solution to the problem (5)-(6)

$$W_n(x,y) = \int_s^y e^{\lambda\tau} \varphi_n(x;\tau) dt + \frac{1}{\beta - 1} \int_S e^{\lambda t} \varphi_n(x;t) dt$$

It is clear that $W_n(x,y)$ is linearly independent.

Indeed, if $\sum_{n=1}^{N} C_n W_n = 0$ is for any set of $W_1, W_2, \dots, W_n$,

then by applying the operator L to this sum, we have

$$\sum_{n=1}^{N} C_n \varphi_n(x;y) = 0 \Rightarrow C_n = 0, \forall n$$

It is clear that $W_n(x;y) \in W_2^1(D)$ is easily obtainable an estimation

$$\|W_n\|_{L_p(D)}^P \le m \|\varphi_n\|_{L_p(D)}^P$$

Moreover, $W_n(x,y)$ satisfies the conditions (6) for any n. We will seek an approximate solution to the problem (1)-(3) in the form of

$$U^N(x;y) = \sum_{n=1}^{N} C_n W_n(x;y)$$

where C_n are constants determined from a system of nonlinear algebraic equations in the form

$$(LU^N_{,} U_n)_0 = (f_{,} U_n)_0, n = 1, N \qquad (7)$$

The solvability of this system of algebraic equations follows from the a priori estimates obtained for the approximate solutions and Lemma « acute angle» from [8]

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Lemma 1. Suppose the conditions $K(x;1)\geq 0$
are satisfied and the inequalities

$$2a(x;y)-K_y(x;y)-\lambda K(x;y)\geq \delta>0$$

Then the estimate holds true

$$\|U^N\|^2_{W_2^1(D)}+\|U^N\|^p_{L_p(D)}\leq K_2 \qquad (8)$$

K_2 does not depend on n.

Proof. Multiplying (7) by C_n and summing over n from 1 to N, we obtain the identity

$$\int_D e^{\lambda y}U_y^N LU^N dD=\int_D e^{\lambda y}U_y^N f dD \qquad (9)$$

Integrating the left-hand side of equation (9) by parts, we obtain

$$\left[\lambda(U_y^N)^2+(2a-\lambda K-Ky)(U_x^N)^2++\lambda(U^N)^2+\frac{2m}{p}\|U^N\|^P\right]dD-$$
$$-\frac{e^\lambda}{2}\int_0^1(U_x^N)^2dx+\frac{e^\lambda}{2}\int_0^1 K(x;1)(U_y^N)^2dx-\frac{e^\lambda}{2}\int_0^1(U^N)^2dx+\frac{1}{2}\int_S e^{\lambda y}\left[((U_y^N)^2-K(U_y^N)^2+m|U^N|^p+(U^N)^2)n_1-2(U_x^N)(U_y^N)n_2\right]ds$$

where $n=(n_1;n_2)$– is the unit vector of the inward normal to ∂D. Using conditions (3) and the conditions of the lemma, we obtain inequality (8). Let's return to the question of the solvability of the system of equations (7). If we write it in the form $\overrightarrow{F_m}(\overrightarrow{C})=0$, where $\vec{C}=(C_{1_m}.....C_{n_m})$ then as we have just seen multiplying $(\overrightarrow{F_m}(\overrightarrow{C}),\overrightarrow{C})_0$ we get the estimate $(\overrightarrow{F_m}(\overrightarrow{C}),\overrightarrow{C})_0\geq K_0\|U^N\|^2_{w_2^1(D)}-K_1$

Since the linear envelope $L(W_1,W_2,......W_m)$ is a finite-dimensional space, there exists $K_2(m)$ such that, therefore, the inequality is satisfied $(\overrightarrow{F_m}(\overrightarrow{C}),\overrightarrow{C})_0\geq K_2(m)\sum_{S=1}^{N}C_S^2-K_1\geq 0$

If $\vec{C}$ is large enough

And this is the "acute angle" condition sufficient for solvability of the system of equations (7).

Theorem. Let the conditions of lemma.

Then for any function $f(x;y)\in L_2(D)$ there exists a generalized solution of the problem (1)–(3).

Proof. By virtue of the estimate (8), the sequence $\left\{\left|U^N\right|^p U^N\right\}$ is bounded in the space L_q where $\frac{1}{p}+\frac{1}{q}=1$. Then, based on (8), from the sequence $\{U^N\}$ we can choose a subsequence converging weakly in $W_2^1(D)$ to some function U(x,y) and the sequence $\left|U^N\right|^p U^N$ converges weakly in $L_q(D)$ to the function $q(x;y)$ $\left|U^N\right|^\rho U^N \to q(x;y)$ in L_q

However, by the theorem, the embedding of $W_2^1(D)$ in $L_2\ (D)$ is quite continuous. Hence, we can assume that the subsequence $U^N(x;y)$ is strong in $L_2\ (D)$ and almost everywhere. Now let us apply lemma 1 of [9],[11] on the limit transition in the nonlinear term in the case where it follows that

$$q(x,y)=|U|^p U$$

Then, passing to the limit at $N\to\infty$ in (7) at fixed n, we have the equality

$$\int_D\left[-U_y(K\varphi_n)_y - U_x\varphi_n + \alpha U\varphi_n + \beta(x;y)U\varphi_n + m|U|^p U\varphi_n\right]dD = \int_D f\varphi_n dD$$

where the function $U(x;y)$ belongs to $W_2^1(D)$. Hence, in view of the density $\{\varphi_n\}$ in the space $W_2^1(D)$ it follows that the integral identity $|4|$ is valid for any $V(x;y)\in \overset{0}{W}{}_2^1(D)$ The theorem is proved.

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